# Nowak's theorem on probability measures induced by strategies revisited

Aiko Kurushima [*], Alexey Piunovskiy [†] and Yi Zhang [‡]

**Abstract:** In this note, we present a new monotone approximation of a given real-valued Carathéodory function on the product $\mathbf{X} \times \mathbf{A}$ of Borel spaces, where $\mathbf{A}$ is also compact. We demonstrate its application by providing a self-contained and elementary proof of a result of A.Nowak in discrete-time Markov decision processes.

**Keywords:** Carathéodory function. Markov decision processes. Monotone approximation.

**AMS 2000 subject classification:** 90C40, 28A20

## 1 Introduction

Let $\mathbf{X}$ is a nonempty Borel space, and $\mathbf{A}$ be a nonempty compact Borel space. A Borel space is regarded as a Borel subset of a Polish space. Throughout we fix the metric $\rho$ in $\mathbf{A}$. Each Borel space is endowed with the Borel $\sigma$-algebra unless stated otherwise, and measurability is always understood in the Borel sense.

An $\mathbb{R}$-valued function $f$ on $\mathbf{X} \times \mathbf{A}$ is called Carathéodory if $f(x,a)$ is measurable in $x \in \mathbf{X}$ for each fixed $a \in \mathbf{A}$, and continuous in $a \in \mathbf{A}$ for each $x \in \mathbf{X}$. It follows that $f$ is jointly measurable on $\mathbf{X} \times \mathbf{A}$, see Lemma 4.15 of [1]. Let the collection of $\mathbb{R}$-valued Carathéodory functions on $\mathbf{X} \times \mathbf{A}$ be denoted by $Car(\mathbf{X} \times \mathbf{A})$.

The approximation of such a Carathéodory function plays a role in studying the space of strategic measures in discrete-time Markov decision processes (DTMDP). In [8], a Carathéodory function was approximated using Luzin or Scorza-Draguni type theorem, which, roughly speaking, asserts that if there is a reference probability measure $\lambda$, then for each $\epsilon > 0$, there is a closed subset $G$ of $\mathbf{X}$ such that $\lambda(G) < \epsilon$ and the restriction of the function $f$ on $G \times \mathbf{A}$ is continuous. Then the Tieze-Dugundij extension theorem extends (continuously) the resulting function to the whole space. This idea was also employed in a more general setup, see [7]. A different approximation was introduced in [11], which we describe as follows. For $f \in Car(\mathbf{X} \times \mathbf{A})$, and each $k = 1, 2, \dots$, consider the $k$th approximation:

$$f_k(x,a) = \sum_{j=1}^{t(k)} f(x, a_j^{(k)}) \varphi_j^{(k)}(a),$$

where $\{a_1^{(k)}, a_2^{(k)}, \dots, a_{t(k)}^{(k)}\}$ is a fixed $1/k$-net of $\mathbf{A}$, $\varphi_j^{(k)}(a) = \frac{\psi_j^{(k)}(a)}{\sum_{i=1}^{t(k)} \psi_i^{(k)}(a)}$, with $\psi_j^{(k)}(a) = \max\{0, 1 - k\rho(a, a_j^{(k)})\}$. Then $f_k(x,a) \to f(x,a)$ uniformly in $a \in \mathbf{A}$ for each $x \in \mathbf{X}$, see p.462 of [11]. Here and

[*]Department of Economics, Sophia University, Tokyo, Japan. E-mail: kurushima@sophia.ac.jp

[†]Department of Mathematical Sciences, University of Liverpool, Liverpool, L69 7ZL, U.K.. E-mail: piunov@liv.ac.uk

[‡]Department of Mathematical Sciences, University of Liverpool, Liverpool, L69 7ZL, U.K.. E-mail: yi.zhang@liv.ac.uk

below, for each $\epsilon > 0$, a finite $\epsilon$-net of the compact Borel space $\mathbf{A}$ is a collection of finitely many points in $\mathbf{A}$ such that for each point $a \in \mathbf{A}$, there is some point $b$ from this collection such that $\rho(a,b) < \epsilon$.

The two approximations are in general not monotone. In this note, we refine the Yushkevich's approximation and provide a monotone nondecreasing approximating sequence, each of whose element is a finite sum of products of a measurable function on $\mathbf{X}$ and a continuous function on $\mathbf{A}$. We state this result in Section 2. In Section 3, we demonstrate the application of this approximation by providing a simple and elementary proof of a famous result of A.Nowak in discrete-time Markov decision processes, where we explain why the original approximation of Yushkevich was not sufficient therein. We finish this note with a conclusion in Section 4.

## 2 Monotone approximation of Carathéodory functions

We now present the following monotone approximation scheme.

**Theorem 2.1** *Let $f \in Car(\boldsymbol{X} \times \boldsymbol{A})$ be fixed. Then there exists a monotone nondecreasing sequence of functions $\{f_k\} \subseteq Car(\boldsymbol{X} \times \boldsymbol{A})$ in the form*

$$f_k(x,a) := \sum_{j=1}^{t(k)} u_j^{(k)}(x) v_j^{(k)}(a)$$

*such that for each $x \in \boldsymbol{X}$, $f_k(x,a) \uparrow f(x,a)$ uniformly in $a \in \boldsymbol{A}$. Here, for each $k = 1, 2, \dots$, $t(k)$ is a finite integer, $u_j^{(k)}$ is measurable on $\boldsymbol{X}$, $v_j^{(k)}$ is $[0,1]$-valued and continuous on $\boldsymbol{A}$. If $f$ is bounded, then one can take $u_j^{(k)}$ bounded on $\boldsymbol{X}$ for each $k = 1, 2, \dots$.*

*Proof.* For each $k = 1, 2, \dots$, we define the following. Let $\{a_1^{(k)}, a_2^{(k)}, \dots, a_{t(k)}^{(k)}\}$ be a finite $\frac{1}{8^k} - 2\frac{1}{8^{k+1}} = \frac{3}{4}\frac{1}{8^k}$-net of the compact metric space $\mathbf{A}$. Let

$$\psi_j^{(k)}(a) := \max\left\{0, 1 - \left(\frac{1}{\frac{1}{8^k} - 2\frac{1}{8^{k+1}}}\right)\rho(a, a_j^{(k)})\right\}$$

with $\rho$ being the metric in $\mathbf{A}$. Let

$$\varphi_j^{(k)}(a) := \frac{\psi_j^{(k)}(a)}{\sum_{i=1}^{t(k)} \psi_i^{(k)}(a)}.$$

So $\varphi_j^{(k)}(a) > 0$ if and only if $\rho(a, a_j^{(k)}) < \frac{1}{8^k} - 2\frac{1}{8^{k+1}}$. Also $\sum_{j=1}^{t(k)} \varphi^{(k)}(a) = 1$. It holds that for each $j = 1, 2, \dots, t(k)$, $\varphi_j^{(k)}$ is $[0,1]$-valued and continuous in $a \in \mathbf{A}$.

Define

$$f_k(x,a) := \sum_{j=1}^{t(k)} \inf_{b \in B(a_j^{(k)}, \frac{1}{8^k})} \{f(x,b)\} \varphi_j^{(k)}(a), \ x \in \mathbf{X}, \ a \in \mathbf{A}, \tag{1}$$

where $B(a, \epsilon)$ is the closed $\epsilon$-ball centered at $a \in \mathbf{A}$. Note that with $\mathbf{A}$ being a compact metric space, $\inf_{b \in B(a_j^{(k)}, \frac{1}{8^k})}\{f(x,b)\}$ is measurable in $x \in \mathbf{X}$ by e.g., Theorem 2 of [6].

In this way, for each fixed $a \in \mathbf{A}$, if $\varphi_j^{(k+1)}(a) > 0$ and $\varphi_i^{(k)}(a) > 0$, then by the triangle inequality, $\rho(a_i^{(k)}, a_j^{(k+1)}) < \frac{1}{8^k} - \frac{1}{8^{k+1}} - \frac{2}{8^{k+2}}$ and

$$B(a_j^{(k+1)}, \frac{1}{8^{k+1}}) \subseteq B(a_i^{(k)}, \frac{1}{8^k}).$$

Therefore, for each fixed $a \in \mathbf{A}$, if $\varphi_j^{(k+1)}(a) > 0$ and $\varphi_i^{(k)}(a) > 0$, then $\inf_{b \in B(a_j^{(k+1)}, \frac{1}{8^{k+1}})}\{f(x,b)\} \geq \inf_{b \in B(a_i^{(k)}, \frac{1}{8^k})}\{f(x,b)\}$.

We verify that the sequence $\{f_k\}$ is monotone nondecreasing. Indeed,

$$\begin{aligned} f_{k+1}(x,a) = & \sum_{j=1}^{t(k+1)} \inf_{b \in B(a_j^{(k+1)}, \frac{1}{8^{k+1}})}\{f(x,b)\}\varphi_j^{(k+1)}(a) \\ \geq & \sum_{j=1}^{t(k+1)} \max_{i:\varphi_i^{(k)}(a)>0}\{\inf_{b \in B(a_i^{(k)}, \frac{1}{8^k})}\{f(x,b)\}\}\varphi_j^{(k+1)}(a) \\ = & \max_{i:\varphi_i^{(k)}(a)>0}\{\inf_{b \in B(a_i^{(k)}, \frac{1}{8^k})}\{f(x,b)\}\} \geq \sum_{i=1}^{t(k)} \inf_{b \in B(a_i^{(k)}, \frac{1}{8^k})}\{f(x,b)\}\varphi_i^{(k)}(a) \\ = & f_k(x,a), \end{aligned}$$

where the second equality and the second inequality follow from the fact that $\sum_{j=1}^{t(k)} \varphi_j^{(k)}(a) = 1$ for each $k$.

Finally, we verify the uniform convergence as follows. Note that

$$|f_k(x,a) - f(x,a)| \leq \sum_{j=1}^{t(k)} \left| \inf_{b \in B(a_j^{(k)}, \frac{1}{8^k})}\{f(x,b)\} - f(x,a) \right| \varphi_j^{(k)}(a)$$

(Again, the fact that $\sum_{j=1}^{t(k)} \varphi_j^{(k)}(a) = 1$ for each $k$ is in use.) For each $j$ such that $\varphi_j^{(k)}(a) > 0$, $a \in B(a_j^{(k)}, \frac{1}{8^k})$. Recall that $f(x,a)$ is uniformly continuous in $a \in \mathbf{A}$ for each fixed $x \in \mathbf{X}$ because $\mathbf{A}$ is a compact metric space. Consequently, for each $\epsilon > 0$, for all sufficiently large $k$, it holds that for each $a \in \mathbf{A}$, $\left|\inf_{b \in B(a_j^{(k)}, \frac{1}{8^k})}\{f(x,b)\} - f(x,a)\right| < \epsilon$ for each $j$ such that $\varphi_j^{(k)}(a) > 0$, and hence $\sup_{a \in \mathbf{A}} |f_k(x,a) - f(x,a)| \leq \epsilon$ for all large enough $k$. The uniform convergence follows.

The proof is completed if one puts $u_j^{(k)}(x) = \inf_{b \in B(a_j^{(k)}, \frac{1}{8^k})}\{f(x,b)\}$ for each $x \in \mathbf{X}$ and $v_j^{(k)}(a) = \varphi_j^{(k)}(a)$ for each $a \in \mathbf{A}$. (If $f$ is bounded, then the additional properties on $u_j^{(k)}$ and $v_j^{(k)}$ are fulfilled automatically.) □

## 3 Alternative proof of Nowak's theorem

We apply Theorem 2.1 in this section to present a more self-contained and elementary proof of a known result due to A.Nowak in the theory of discrete-time Markov decision processes. A special case was proved in [11], which did not require the monotone approximation of a Carathéodory function of the type in Theorem 2.1, which is needed in order to generalize the reasoning in [11].

We describe a DTMDP as follows, see more details in [5, 9]. Let $S$ be the nonempty Borel state space, $A$ be the nonempty Borel action space, $p$ be a stochastic kernel from $S \times A$ to $\mathcal{B}(S)$, and $\nu$ be the initial distribution on $\mathcal{B}(S)$. Let us denote for each $n = 1, 2, \ldots, \infty$, $\mathbf{H}_n := S \times (A \times S)^n$ and $\mathbf{H}_0 := S$.

A strategy $\sigma = \{\sigma_n\}_{n=0}^{\infty}$ in the DTMDP is given by a sequence of stochastic kernels $\sigma_n(da|h_n)$ on $\mathcal{B}(A)$ from $h_n \in \mathbf{H}_n$ for $n = 0, 1, 2, \ldots$. Let $\Sigma$ be the space of strategies. Let the controlled and controlling processes be denoted by $\{X_n\}_{n=0}^{\infty}$ and $\{A_n\}_{n=0}^{\infty}$. Here, for each $n = 0, 1, \ldots$, $X_n$ (respectively, $A_n$) is the projection of $\mathbf{H}_\infty$ to the $2n+1$st (respectively, the $2n+2$nd) coordinate.

Under a strategy $\sigma = \{\sigma_n\}$, by the Ionescu-Tulcea theorem, one can construct a probability measure $\mathbf{P}_\nu^\sigma$ on $(\mathbf{H}_\infty, \mathcal{B}(\mathbf{H}_\infty))$ such that

$$\begin{aligned}
&\mathbf{P}_\nu^\sigma(X_0 \in dx) = \nu(dx), \\
&\mathbf{P}_\nu^\sigma(A_n \in da|X_0, A_0, \dots, X_n) = \sigma_n(da|X_0, A_0, \dots, X_n),\ n = 0, 1, \dots, \\
&\mathbf{P}_\nu^\sigma(X_{n+1} \in dx|X_0, A_0, \dots, X_n, A_n) = p(dx|X_n, A_n),\ n = 0, 1, \dots.
\end{aligned}$$

As usual, equalities involving conditional expectations and probabilities are understood in the almost sure sense. The probability measure $\mathbf{P}_\nu^\sigma$ is called a strategic measure for the DTMDP (under the strategy $\sigma$). Let $\mathcal{S} := \{\mathbf{P}_\nu^\sigma : \ \sigma \in \Sigma\}$ be the space of all strategic measures for the DTMDP (with the fixed initial distribution $\nu$ throughout this note).

Let the space of probability measures on $(\mathbf{H}_\infty, \mathcal{B}(\mathbf{H}_\infty))$ be denoted by $\mathbb{P}(\mathbf{H}_\infty)$. The $w$-topology on $\mathbb{P}(\mathbf{H}_\infty)$ is the weakest topology with respect to which, for each $n = 1, 2, \dots$, $\int_{\mathbf{H}_n} f(h_n)\mu(dh_n \times (A \times S)^\infty)$ is continuous in $\mu \in \mathbb{P}(\mathbf{H}_\infty)$ for each bounded continuous function $f$ on $\mathbf{H}_n$. This is the same as to say that the $w$-topology is generated by the class of bounded continuous functions on $\mathbf{H}_\infty$, see Lemma 4.1 of [10]. Similarly, the $ws$-topology is the weakest topology with respect to which, for each $n = 1, 2, \dots$, $\int_{\mathbf{H}_n} f(h_n)\mu(dh_n \times (A \times S)^\infty)$ is continuous in $\mu \in \mathbb{P}(\mathbf{H}_\infty)$ for each bounded measurable function $f$ on $\mathbf{H}_n$ such that $f(x_0, a_0, x_1, a_1, \dots, a_{n-1}, x_n)$ is continuous in $(a_0, a_1, \dots, a_{n-1}) \in A^n$ (keeping the other arguments fixed). The $w$-topology on $\mathbb{P}(\mathbf{H}_\infty)$ is strictly weaker than the $ws$-topology in general. Nevertheless, A.Nowak noted that under some conditions, the two topologies are equivalent when they are restricted to $\mathcal{S} \subseteq \mathbb{P}(\mathbf{H}_\infty)$, see Theorem 1 of [8], which we quote as follows.

**Theorem 3.1 (A.Nowak)** *Suppose that the action space $A$ is compact, and for each bounded measurable function $f$ on $S$, the integral $\int_S f(y)p(dy|x, a)$ is continuous in $a \in A$ for each $x \in S$. Then, restricted to the space of strategic measures $\mathcal{S}$, the (relative) $w$-topology coincides with the (relative) $ws$-topology.*

*Proof.* Let $\{\mu_\alpha\} \subseteq \mathcal{S}$ be a net such that $\mu_\alpha \to \mu \in \mathcal{S}$ in the $w$-topology. It suffices to show that $\mu_\alpha \to \mu \in \mathcal{S}$ in the $ws$-topology, see p.127 of [4]. Let $n = 1, 2, \dots$ be fixed. Let us show that

$$\int_{\mathbf{H}_n} f(h_n)\mu_\alpha(dh_n \times (A \times S)^\infty) \to \int_{\mathbf{H}_n} f(h_n)\mu(dh_n \times (A \times S)^\infty) \tag{2}$$

for each bounded measurable function $f$ on $\mathbf{H}_n$ satisfying $f(x_0, a_0, x_1, a_1, \dots, a_{n-1}, x_n)$ is continuous in $(a_0, a_1, \dots, a_{n-1}) \in A^n$ (keeping the other arguments fixed).

Firstly, we verify that (2) holds for each $f$ in the form of $f(h_n) = d(x_0, x_1, \dots, x_n)g(a_0, a_1, \dots, a_{n-1})$, where $d$ is a bounded measurable function on $S^{n+1}$, and $g$ is a bounded continuous function on $A^n$, as follows. Let some bounded continuous function $g$ on $A^n$ be arbitrarily fixed, and $\mathcal{D}$ be the set of bounded measurable functions $d$ on $S^{n+1}$ such that

$$\begin{aligned}
&\int_{\mathbf{H}_n} d(x_0, x_1, \dots, x_n)g(a_0, a_1, \dots, a_{n-1})\mu_\alpha(dh_n \times (A \times S)^\infty) \\
\to &\int_{\mathbf{H}_n} d(x_0, x_1, \dots, x_n)g(a_0, a_1, \dots, a_{n-1})\mu(dh_n \times (A \times S)^\infty).
\end{aligned}$$

Then $\mathcal{D}$ is a vector space, and contains real constants. Now let $\{d_m\}$ be a monotone nondecreasing and uniformly bounded sequence of nonnegative functions in $\mathcal{D}$ such that $d_m \uparrow d_\infty$ pointwise. Then $d_\infty$ is bounded measurable. To show that $d_\infty$ is in $\mathcal{D}$, we introduce the following notations. For each $k = 1, 2, \dots$, we introduce an operator say $T_k$ mapping a bounded measurable function say $u$ on $\mathbf{H}^k$

such that it is continuous on $A^k$ (keeping the other arguments fixed) to a bounded measurable function on $\mathbf{H}_{k-1}$ such that it is continuous on $A^{k-1}$ (keeping the other arguments fixed), defined by

$$T_k \circ u(x_0, a_0, x_1, a_1, \ldots, a_{k-2}, x_{k-1}) := \sup_{a_{k-1}\in A} \int_S u(x_0, a_0, x_1, a_1 \ldots, a_{k-1}, x_k) p(dx_k|x_{k-1}, a_{k-1}),$$

where the right hand side indeed defines a measurable bounded function on $\mathbf{H}^{k-1}$ continuous on $A^{k-1}$ (keeping the other arguments fixed) by Theorem 2 of [6], Proposition 7.30 of [2] and the Berge theorem, see e.g., Theorem 17.31 of [1], which are applicable under the conditions of this theorem. In particular, $T_k$ maps a bounded measurable function on $S^{k+1}$ to a bounded measurable function on $S^k$.

Now,

$$\begin{aligned}
&\left|\int_{\mathbf{H}_n} d_m(x_0, x_1, \ldots, x_n) g(a_0, a_1, \ldots, a_{n-1}) \mu_\alpha(dh_n \times (A \times S)^\infty)\right.\\
&\left.-\int_{\mathbf{H}_n} d_\infty(x_0, x_1, \ldots, x_n) g(a_0, a_1, \ldots, a_{n-1}) \mu_\alpha(dh_n \times (A \times S)^\infty)\right|\\
\leq\ & \sup_{(a_0,a_1,\ldots,a_{n-1})\in A^n} |g|(a_0, a_1, \ldots, a_{n-1}) \int_{\mathbf{H}_n} |d_m - d_\infty|(x_0, x_1, \ldots, x_n) \mu_\alpha(dh_n \times (A \times S)^\infty)\\
\leq\ & \sup_{(a_0,a_1,\ldots,a_{n-1})\in A^n} |g|(a_0, a_1, \ldots, a_{n-1}) \int_S T \circ T_2 \circ \cdots \circ T_{n-1} \circ T_n \circ |d_m - d_\infty|(x) \nu(dx).
\end{aligned}$$

Here and below, given a $\mathbb{R}$-valued function $g$, the notation $|g|(\cdot) := |g(\cdot)|$ is in use. Note that by the dominated convergence theorem,

$$\begin{aligned}
&\lim_{m\to\infty} \int_S T \circ T_2 \circ \cdots \circ T_{n-1} \circ T_n \circ |d_m - d_\infty|(x_0) \nu(dx_0)\\
=\ & \int_S \lim_{m\to\infty} \sup_{a_0\in A} \int_S T_2 \circ \cdots \circ T_{n-1} \circ T_n \circ |d_m - d_\infty|(x_0, x_1) p(dx_1|x_0, a_0) \nu(dx_0)
\end{aligned}$$

Since $\int_S T_2 \circ \cdots \circ T_{n-1} \circ T_n \circ |d_m - d_\infty|(x_0, x_1) p(dx_1|x_0, a_0)$ is monotone nonincreasing in $m$ and continuous in $a_0 \in A$ under the conditions of the statement, the order of $\sup_{a_0\in A}$ and $\lim_{m\to\infty}$ can be interchanged, so that

$$\begin{aligned}
&\lim_{m\to\infty} \int_S T \circ T_2 \circ \cdots \circ T_{n-1} \circ T_n \circ |d_m - d_\infty|(x_0) \nu(dx_0)\\
=\ & \int_S \sup_{a_0\in A} \lim_{m\to\infty} \int_S T_2 \circ \cdots \circ T_{n-1} \circ T_n \circ |d_m - d_\infty|(x_0, x_1) p(dx_1|x_0, a_0) \nu(dx_0).
\end{aligned}$$

Thus, an iterative argument reveals

$$\lim_{m\to\infty} \int_S T \circ T_2 \circ \cdots \circ T_{n-1} \circ T_n \circ |d_m - d_\infty|(x_0) \nu(dx_0) = 0.$$

That is, the convergence

$$\begin{aligned}
&\int_{\mathbf{H}_n} d_m(x_0, x_1, \ldots, x_n) g(a_0, a_1, \ldots, a_{n-1}) \mu_\alpha(dh_n \times (A \times S)^\infty)\\
\to\ & \int_{\mathbf{H}_n} d_\infty(x_0, x_1, \ldots, x_n) g(a_0, a_1, \ldots, a_{n-1}) \mu_\alpha(dh_n \times (A \times S)^\infty)
\end{aligned}$$

is uniform with respect to $\alpha$. Consequently, by Lemma 6 in p.28 of [3],

$$\begin{aligned}
&\lim_m \lim_\alpha \int_{\mathbf{H}_n} d_m(x_0, x_1, \ldots, x_n) g(a_0, a_1, \ldots, a_{n-1}) \mu_\alpha(dh_n \times (A \times S)^\infty) \\
= \; &\lim_\alpha \lim_m \int_{\mathbf{H}_n} d_m(x_0, x_1, \ldots, x_n) g(a_0, a_1, \ldots, a_{n-1}) \mu_\alpha(dh_n \times (A \times S)^\infty),
\end{aligned}$$

By the dominated convergence theorem, this implies

$$\begin{aligned}
&\int_{\mathbf{H}_n} d_\infty(x_0, x_1, \ldots, x_n) g(a_0, a_1, \ldots, a_{n-1}) \mu(dh_n \times (A \times S)^\infty) \\
= \; &\lim_\alpha \int_{\mathbf{H}_n} d_\infty(x_0, x_1, \ldots, x_n) g(a_0, a_1, \ldots, a_{n-1}) \mu_\alpha(dh_n \times (A \times S)^\infty),
\end{aligned}$$

i.e., $d_\infty \in \mathcal{D}$. Since $\mu_\alpha \to \mu$ in the $w$-topology by assumption, the class $\mathbb{C}(S^{n+1})$ of all bounded continuous functions on $S^{n+1}$, which is closed under multiplication, is in $\mathcal{D}$. Thus, by the functional monotone class theorem, $\mathcal{D}$ contains all bounded functions measurable with respect to the $\sigma$-algebra generated by $\mathbb{C}(S^{n+1})$, which is $\mathcal{B}(S^{n+1})$. Since the continuous bounded function $g$ on $A^n$ was arbitrarily fixed, this shows that (2) holds for each $f$ in the form of $f(h_n) = d(x_0, x_1, \ldots, x_n) g(a_0, a_1, \ldots, a_{n-1})$, where $d$ is a bounded measurable function on $S^{n+1}$, and $g$ is a bounded continuous function on $A^n$, as desired.

Secondly, we show (2) holds for each bounded measurable function $f$ on $\mathbf{H}_n$ such that $f(h_n)$ is continuous in $(a_0, a_1, \ldots, a_{n-1}) \in A^n$ (keeping the other arguments fixed), as follows. Let such a function $f$ be fixed. Note that $f$ can be regarded as an element of $Car(S^{n+1} \times A^n)$. Take a monotone nondecreasing approximating sequence $\{f_m\}$ of $f$ that comes from Theorem 2.1. Then from what was established in the first step, (2) holds for each $f_m$. Now

$$\begin{aligned}
&\left| \int_{\mathbf{H}_n} f_m(h_n) \mu_\alpha(dh_n \times (A \times S)^\infty) - \int_{\mathbf{H}_n} f(h_n) \mu_\alpha(dh_n \times (A \times S)^\infty) \right| \\
\leq \; &\int_{\mathbf{H}_n} |f_m - f|(h_n) \mu_\alpha(dh_n \times (A \times S)^\infty) \leq \int_S T \circ T_2 \circ \cdots \circ T_{n-1} \circ T_n \circ |f_m - f|(x) \nu(dx).
\end{aligned}$$

As in Step 1,

$$\begin{aligned}
&\lim_{m \to \infty} \int_S T \circ T_2 \circ \cdots \circ T_{n-1} \circ T_n \circ |f_m - f|(x_0) \nu(dx_0) \\
= \; &\int_S \lim_{m \to \infty} \sup_{a_0 \in A} \int_S T_2 \circ \cdots \circ T_{n-1} \circ T_n \circ |f_m - f|(x_0, a_0, x_1) p(dx_1 | x_0, a_0) \nu(dx_0) \\
= \; &\int_S \sup_{a_0 \in A} \lim_{m \to \infty} \int_S T_2 \circ \cdots \circ T_{n-1} \circ T_n \circ |f_m - f|(x_0, a_0, x_1) p(dx_1 | x_0, a_0) \nu(dx_0), \qquad (3)
\end{aligned}$$

where the last equality holds since $\int_S T_2 \circ \cdots \circ T_{n-1} \circ T_n \circ |f_m - f|(x_0, a_0, x_1) p(dx_1 | x_0, a_0)$ is monotone nonincreasing in $m$ and continuous in $a_0 \in A$ under the conditions of the statement. The rest proceeds as in Step 1. Eventually,

$$\lim_m \lim_\alpha \int_{\mathbf{H}_n} f_m(h_n) \mu_\alpha(dh_n \times (A \times S)^\infty) = \lim_\alpha \lim_m \int_{\mathbf{H}_n} f_m(h_n) \mu_\alpha(dh_n \times (A \times S)^\infty),$$

and hence

$$\int_{\mathbf{H}_n} f(h_n) \mu(dh_n \times (A \times S)^\infty) = \lim_\alpha \int_{\mathbf{H}_n} f(h_n) \mu_\alpha(dh_n \times (A \times S)^\infty).$$

The statement is proved. □

As a corollary of the above result, under the conditions therein, the (relative) $ws$-topology on $\mathcal{S}$ is seperable and metrizable, and $\mathcal{S}$, endowed with the corresponding Borel $\sigma$-algebra, is a Borel space.

We emphasize that in the above proof of Theorem 3.1 for the last equality in (3), it is needed that the approximating sequence $\{f_m\}$ to $f \in Car(S^{n+1} \times A^n)$ is monotone, because in general one cannot interchange the order of superemum and limit.

The idea of the above proof is basically from [11], where the author showed the following special case. Let $\Gamma$ be the space of stochastic kernels from $\mathbf{X}$ to $\mathbf{A}$. Though the author dealt with sequences, the reasoning in the proof of Theorem 1 in [11] actually also shows that the (relative) $ws$-topology on $\{\mu \in \mathbb{P}(S \times A): \ \mu(dx \times da) = \nu(dx)\sigma(da|x), \ \sigma \in \Gamma\}$ is equivalent to the $w$-topology on it. Since the model is zero-step, there was no need for the monotone approximation given in Theorem 2.1.

Finally, the proof of Theorem 3.1 here is elementary and simple. In comparison, the proof in [8] is based on a different reasoning and less self-contained, e.g., it refers to Lemma 6.2 in [10], which was in turn proved using the facts from non-stationary dynamic programming.

# 4 Conclusion

In this note, we provided a monotone approximating sequence of a given real-valued Carathéodory function on the product $\mathbf{X} \times \mathbf{A}$ of Borel spaces, where $\mathbf{A}$ is also compact. Each element in this sequence is a finite sum of products of a measurable function on $\mathbf{X}$ and a continuous function on $\mathbf{A}$. We demonstrate its application by providing a self-contained and elementary proof of a famous result of A.Nowak in discrete-time Markov decision processes.

**Acknowledgement.** This work is partially supported by a grant from the Royal Society (IE160503). We are grateful to Professor Andrzej Nowak for kindly sending us his paper [8].

# References

[1] Aliprantis, C. and Border, K. (2007). *Infinite Dimensional Analysis.* Springer, New York.

[2] Bertsekas, D. and Shreve, S. (1978). *Stochastic Optimal Control.* Academic Press, New York.

[3] Dunford, N. and Schwartz, J. (1958). *Linear Operators. Part I: General Theory.* Interscience Publishers, New York.

[4] Gemignani, M. (1990). *Elementary Topology.* Dover, New York.

[5] Hernández-Lerma, O. and Lasserre, J. (1996). *Discrete-Time Markov Control Processes.* Springer-Verlag, New York.

[6] Himmelberg, C., Parthasarathy, T., and Van Vleck, F. (1976). Optimal plans for dynamic programming problems. *Math. Oper. Res.* **1**, 390-394.

[7] Kubińska, E. (2004). Approximation of Carathéodory functions and multifunctions. *Real Analysis Exchange* **30**, 251-360.

[8] Nowak, A. (1988). On the weak topology on a space of probability measures induced by policies. *Bulletin of Polish Academy of Sciences (Mathemataics)* **36**, 182-186.

[9] Piunovskiy, A. (1997). *Optimal Control of Random Sequences in Problems with Constraints.* Kluwer, Dordrecht.

[10] Schäl, M. (1975). On dynamic programming: compactness of the space of policies. *Stoch. Proc. Appl.* **3**, 345-364.

[11] Yushkevich, A. (1997). The compactness of a policy space in dynamic programming via an extension theorem for Carathéodory functions. *Math. Oper. Res.* **22**, 458-467.